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EE 364 - Spring 2026

Week 10
Supplemental Notes

Topics

1. Multiple random variables
2. Uncertainty Principle

$$\sigma_{XY}^2 \leq \sigma_X^2 \sigma_Y^2$$

3. Sample vs. Population

Multiple random variables

Borel σ-algebra $(\mathbb{R}^n)$

$$\mathcal{B}(\mathbb{R}^n) = \sigma\left(\left\{ (-\infty, a_1] \times (-\infty, a_2] \times \cdots \times (-\infty, a_n] : a_k \in \mathbb{R} \text{ for } k = 1, \ldots, n \right\} \right)$$

random variable: $(\Omega, \mathcal{A}, P)$ $\quad X : \Omega \longrightarrow \mathbb{R}^n$

PAM: $\forall \Theta \in \mathcal{B}(\mathbb{R}^n), \; X^{-1}(\Theta) \in \mathcal{A}$

recall: If $X^{-1}(\Theta) \notin \mathcal{A}$ can't put probability on it

n-dimensional CDF / pdf

2-D: $F_{XY}(x,y) = P(\{X \le x \text{ AND } Y \le y\})$

$= P(\{\omega \in \Omega : X(\omega) \le x \text{ AND } Y(\omega) \le y\})$

Absolute continuity $\quad f_{XY}(x,y) = \dfrac{\partial^2 F_{XY}}{\partial x \, \partial y}$

$$F_{XY}(x,y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f_{XY}(u,v) \, dv \, du$$

n-D: $F(x_1, x_2, \dots, x_n) = P(\{X_1 \le x_1 \text{ AND } X_2 \le x_2 \dots \text{ AND } X_n \le x_n\})$

$$f(x_1, \dots, x_n) = \frac{\partial^n F}{\partial x_1 \cdots \partial x_n}$$

Properties of joint CDF / pdf

- $F_{XY}(x, -\infty) = 0 = F_{XY}(-\infty, y)$
- $F_{XY}(x, \infty) = \int_{u=-\infty}^{u=x} \int_{v=-\infty}^{v=\infty} f_{XY}(u,v) \, dv \, du = \int_{-\infty}^{x} f_X(u) \, du. = F_X(x)$

marginal CDF

"integrate out y" $\longrightarrow$ marginal for X

- continuous

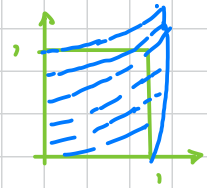
$f_{XY}(x,y) \cdot dx\,dy \ge 0 \qquad \iint_{\mathbb{R}} f_{XY}(x,y) \, dy \, dx = 1$

discrete

$p_{XY}(x,y) \ge 0 \qquad \sum_x \sum_y p_{XY}(x,y) = 1$

(i.e. p_{XY} are "convex coefficients")

Ex: $f_{XY}(x,y) = 6x^2y$ if $0 \le x \le 1$, $0 \le y \le 1$

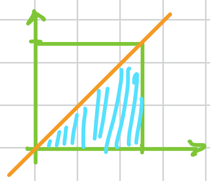


$$f_X(x) = \int_{y=-\infty}^{y=\infty} f_{XY}(x,y)\,dy = \int_0^1 6x^2y\,dy$$
$$= 6x^2 \int_0^1 y\,dy$$
$$= 6x^2 \left.\frac{y^2}{2}\right|_{y=0}^{y=1}$$
$$= 3x^2 \quad \text{if} \quad 0 \le x \le 1$$

$$f_Y(y) = \int_{x=-\infty}^{x=\infty} f_{XY}(x,y)\,dx = \cdots = 2y \quad \text{if} \quad 0 \le y \le 1.$$

note: $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

$$P(X > Y) = \int_{x=0}^{x=1}\int_{y=0}^{y=x} 6x^2y\,dy\,dx = 6\int_{x=0}^{x=1} x^2 \cdot \left.\frac{y^2}{2}\right|_{y=0}^{y=x} dx$$



$$= 3\int_{x=0}^{x=1} x^4\,dx = 3 \cdot \left.\frac{x^5}{5}\right|_{x=0}^{x=1} = \frac{3}{5}$$

$$\therefore\ f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{6x^2y}{3x^2} = 2y$$

note: $= f_Y(y)$

joint pdf $\longleftrightarrow$ $f_{XY}(x,y)$

marginal pdf $\longleftrightarrow$ $f_X(x)$ and $f_Y(y)$

conditional pdf $\longleftrightarrow$ $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$ $\quad f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

Defn: Random ("stochastic") process, family of random variables $\{Y_\alpha\}_{\alpha \in I}$

index set.

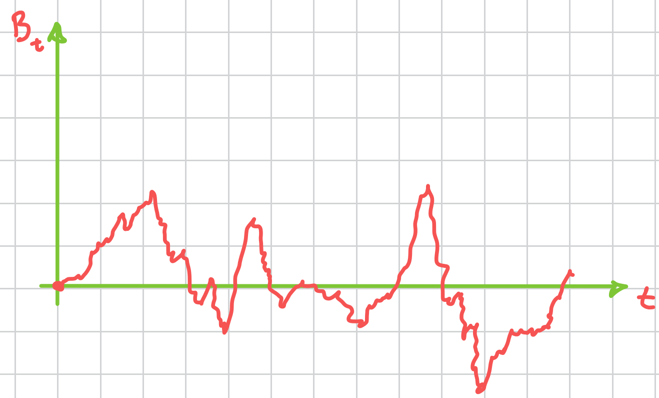
- scalar r.v $\quad X, \quad I = \{1\}$
- vector r.v. $\quad \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}, \quad I = \{1, \dots, n\}$
- sequence r.v. $\quad X_1, X_2, \dots, \quad I = \mathbb{Z}^+$
- random process $\quad \{X_t\}, \quad t \in \mathbb{R}$
- random field $\quad \{X_{uv}\}$ image or "pixels"

  $\{X_{uvw}\}$ volume or "voxels"

Ex: Brownian motion

1. $B_0 = 0$
2. $B_t - B_s \sim N(0, s-t)$
3. independent increments

   $B_t - B_s \quad \forall t, s$



Independence

Defn: random variables X and Y are independent

$\longleftrightarrow$ "joint factors"

$$F_{XY}(x,y) = P(\{X \le x \text{ AND } Y \le y\}) = P(\{X \le x\}) \cdot P(\{Y \le y\}) = F_X(x) \cdot F_Y(y) \quad \forall x, y.$$

$$\longleftrightarrow \quad f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

for n-dimensions:

$$f(x_1, x_2, \dots, x_n) = \prod_{k=1}^{n} f_{X_k}(x_k)$$

later, to optimize take log before derivative

i.e, $\log f(x_1, \dots, x_n) = \sum_{k=1}^{n} \log f_{X_k}(x_k)$

$$X_1, X_2, \dots, X_n \text{ i.i.d.} \longleftrightarrow f(x_1, \dots, x_n) \overset{\text{indep.}}{=} \prod_{k=1}^{n} f_{X_k}(x_k) \overset{\text{ident.}}{=} [f_X(x)]^n$$

Ex: "Similarly distributed" $X_k \sim \text{Poisson}\left(\frac{1}{k}\right)$, $Y_k \sim N\left(0, \frac{1}{k^2}\right)$ same pdf, just different parameters.

★★★ You can't prove independence in the real world. You assume it!

must argue for it

Moments of Multiple Random Variables

Defn: The joint expected value

$$E_{XY}\left[g(X,Y)\right] \overset{\text{continuous}}{=} \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} g(x,y)\, f_{XY}(x,y)\, dy\, dx$$

$$\overset{\text{discrete}}{=} \sum_x \sum_y g(x,y) \cdot p_{XY}(x,y)$$

if $E\left[\,|g(X,Y)|\,\right] < \infty$

Thrm:

① $E_{XY}\left[aX + bY + c\right] = a \cdot E_X[X] + b \cdot E_Y[Y] + c$

② $V_{XY}\left[aX + bY + c\right] = a^2 \cdot V_X[X] + b^2 \cdot V_Y[Y] + 2ab \cdot \sigma_{XY}$

Prf: ① continuous case

$$E_{XY}\left[aX + bY + c\right] = \int_x \int_y (ax + by + c)\, f_{XY}(x,y)\, dy\, dx$$

$$= a \cdot \int_x x \underbrace{\left(\int_y f_{XY}(x,y)\, dy \right)}_{= f_X} dx$$

$$+\; b \cdot \int_y y \underbrace{\left(\int_x f_{XY}(x,y)\, dx \right)}_{= f_Y} dy$$

$$+\; c \underbrace{\int_x \int_y f_{XY}(x,y)\, dy\, dx}_{= 1}$$

$$= \underbrace{a \cdot \int_x x \cdot f_X(x)\, dx}_{= E[X]} + \underbrace{b \cdot \int_y y \cdot f_Y(y)\, dy}_{= E[Y]} + c$$

$$= a \cdot E_X[X] + b \cdot E_Y[Y] + c$$

QED

② $V_{XY}[aX+bY+c] = E_{XY}\left[\left(aX+bY+c - E[aX+bY+c]\right)^2\right]$

$$= E_{XY}\left[\left(aX+bY+\cancel{c} - a\cdot E[X] - b\cdot E[Y] - \cancel{c}\right)^2\right]$$

$$= E_{XY}\left[\left(a(X-E_X[X]) + b(Y-E_Y[Y])\right)^2\right]$$

$$= E_{XY}\left[a^2\cdot(X-E_X[X])^2 + b^2\cdot(Y-E_Y[Y])^2 + 2ab\,(X-E_X[X])(Y-E_Y[Y])\right]$$

$$= a^2\cdot E_{XY}[(X-E_X[X])^2] + b^2\cdot E_{XY}[(Y-E_Y[Y])^2] + 2ab\,E_{XY}[(X-E_X[X])(Y-E_Y[Y])]$$

$$= a^2\cdot \underbrace{E_X[(X-E_X[X])^2]}_{=V_X[X]} + b^2\cdot \underbrace{E_Y[(Y-E_Y[Y])^2]}_{=V_Y[Y]} + 2ab\,\underbrace{E_{XY}[(X-E_X[X])(Y-E_Y[Y])]}_{=\sigma_{XY}}$$

$$= a^2\cdot V_X[X] + b^2\cdot V_Y[Y] + 2ab\cdot\sigma_{XY}$$

Defn: The correlation of X and Y: $E_{XY}[XY]$

"averages factor"

Defn: X and Y uncorrelated $\longleftrightarrow$ $E_{XY}[XY] = E_X[X]\cdot E_Y[Y]$

(notation: $\perp$ $\longleftrightarrow$ $E_{XY}[XY]=0$)

"orthogonal"

Ex: $X \sim U[-1,1]$

$Y = X^2$

$\therefore$ X and Y are not independent.

$$\therefore\ E_{XY}[XY] = E_{XY}[X\cdot X^2] = E_{XY}[X^3] = E_X[X^3] = \int_{-\infty}^{\infty} x^3\cdot f_X(x)\,dx$$

$$= \int_{-1}^{1} x^3\cdot\frac{1}{2}\,dx = \frac{1}{2}\,\frac{x^4}{4}\Big|_{x=-1}^{x=1} = 0$$

$\therefore$ uncorrelated but not independent.

Defn: The covariance of X and Y:

$$\sigma_{XY} \triangleq \mathrm{Cov}(X,Y)$$
$$= E_{XY}\left[(X-\mu_X)(Y-\mu_Y)\right] \quad \text{(if exists)}$$

E for joint pdf $f_{XY}(x,y)$

Corr: X and Y uncorrelated $\quad V[X+Y] = V[X] + V[Y]$

Thrm: Independent ($\therefore$ uncorrelated) $X_1, \dots, X_n \longrightarrow$

$$V\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} V[X_k]$$

Thrm: $V\left[\sum_{k=1}^{n} X_k\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} \longleftarrow$ note: $\sigma_{ii} = E[(X_i - E[X_i])(X_i - E[X_i])]$

$$= E[(X_i - E[X_i])^2]$$
$$= V[X_i]$$

$$= \underbrace{\sum_{i=1}^{n} V[X_i]}_{n\text{-terms}} + \underbrace{\sum_{\substack{i=1 \\ i \neq j}}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j)}_{n(n-1)\text{ terms}}$$

Corr: Covariance is "bilinear" $\quad \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, X_j)$

Corr: if X and Y independent: $V[XY] = V[X]\cdot V[Y] + V[X]\cdot E^2[Y] + E^2[X]\cdot V[Y]$

Thrm: $\sigma_{XY} = \mathrm{Cov}(X,Y) = E_{XY}[XY] - E_X[X]\cdot E_Y[Y]$

Prf: $\sigma_{XY} = E_{XY}[(X-E[X])(Y-E[Y])] = E_{XY}[XY - X\cdot E[Y] - E[X]\cdot Y + E[X]\,E[Y]]$

$$= E_{XY}[XY] - E_{XY}[X\cdot E[Y]] - E_{XY}[E[X]\cdot Y] + E_{XY}[E[X]\cdot E[Y]]$$

constant

$$= E_{XY}[XY] - E_X[X]\cdot E_Y[Y] - \cancel{E_X[X]}\,\cancel{E_Y[Y]} + \cancel{E_X[X]\,E_Y[Y]}$$

$$= E_{XY}[XY] - E_X[X]\cdot E_Y[Y]$$

Corr: If X and Y uncorrelated $\longrightarrow$ $\sigma_{XY} = 0$

Corr: If X and Y independent $\longrightarrow$ X and Y are uncorrelated (i.e, $\sigma_{XY} = 0$)

note: $\cancel{\longleftarrow}$ in general

Prf:

$$E_{XY}[XY] = \int_x \int_y x \cdot y \cdot f_{XY}(x,y)\, dy\, dx \overset{\text{indep.}}{=} \int_x \int_y xy \cdot f_X(x) \cdot f_Y(y)\, dy\, dx.$$

$$= \Big(\underbrace{\int_x x\, f_X(x)\, dx}_{= E_X[X]} \Big) \Big(\underbrace{\int_y y \cdot f_Y(y)\, dy}_{= E_Y[Y]} \Big)$$

$$= E_X[X] \cdot E_Y[Y]$$

Note: correlated data cannot be independent

Thm: (Cauchy Schwartz Uncertainty Principle - U.P.)

$$\sigma_{XY}^2 \le \sigma_X^2 \sigma_Y^2$$

Prf: $\forall c \in \mathbb{R}$

$$0 \le E_{XY}\left[\left((X-\mu_X) - c\cdot(Y-\mu_Y)\right)^2\right]$$

$$= E_{XY}\left[(X-\mu_X)^2 + c^2\cdot(Y-\mu_Y)^2 - 2c\cdot(X-\mu_X)(Y-\mu_Y)\right]$$

$$= E_{XY}\left[(X-\mu_X)^2\right] + c^2\cdot E_{XY}\left[(Y-\mu_Y)^2\right] - 2c\cdot E_{XY}\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$= E_X\left[(X-\mu_X)^2\right] + c^2\cdot E_Y\left[(Y-\mu_Y)^2\right] - 2c\,\sigma_{XY}$$

$$= \sigma_X^2 + c^2\cdot\sigma_Y^2 - 2c\cdot\sigma_{XY}$$

$\therefore$ put $c = \frac{\sigma_{XY}}{\sigma_Y^2}$ since holds $\forall c \in \mathbb{R}$.

$$= \sigma_X^2 + \frac{\sigma_{XY}^2}{\sigma_Y^4}\cdot\sigma_Y^2 - 2\cdot\frac{\sigma_{XY}}{\sigma_Y^2}\cdot\sigma_{XY}$$

$$= \sigma_X^2 + \frac{\sigma_{XY}^2}{\sigma_Y^2} - 2\cdot\frac{\sigma_{XY}^2}{\sigma_Y^2}$$

$$= \sigma_X^2 - \frac{\sigma_{XY}^2}{\sigma_Y^2}$$

$\therefore \frac{\sigma_{XY}^2}{\sigma_Y^2} \le \sigma_X^2$ $\qquad \therefore \sigma_{XY}^2 \le \sigma_X^2\sigma_Y^2$

$\therefore \quad |\sigma_{XY}| \overset{\text{u.p.}}{\leq} \sigma_X \sigma_Y$

$\therefore \quad -\sigma_X \sigma_Y \leq \sigma_{XY} \leq \sigma_X \sigma_Y$

$\therefore \quad -1 \leq \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y} \leq 1$

Defn: The population correlation coefficient:

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$\therefore \quad -1 \leq \rho_{XY} \leq 1$